Integrals of motion with displacement transformations

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Brief outline

- The model
- Diagonalization of the Hamiltonian through displacement transformations
- Properties of the diagonalized Hamiltonian and of the integrals of motion
- Results

Many-body localization

- With strong enough disorder, all single-particle states are localized.
- Conductivity is then assumed to be by hopping between localized states. Mott's variable range hopping.
- The standard driving mechanism for hopping is the phonon bath, but any extended, continuous bath could do the same role.
- Basko et al. (2006) proposed the electron-electron interaction as the driving mechanism above a certain temperature.
- The problem can be thought of as many-body localization in Fock space.

Model

- Spinless fermions in 1D
- The total Hamiltonian that we consider is

$$H = \sum_i \epsilon_i n_i + \sum_{\langle i,j \rangle} t c_j^\dagger c_i + \frac{1}{2} \sum_{\langle i,j \rangle} V n_j n_i$$

  where $c_i^\dagger$ is the creation operator on site i and $n_i = c_i^\dagger c_i$
- $t = 1$ is our unit of energy and $\epsilon_i \in [-W/2, W/2]$
- Short range potential: nearest neighbours for fermions ($V = 1$)

Previous simulations

- List of talks of this conference
- Exact diagonalization
- DMRG
- Local Integrals of Motion
  - Huse, Nandkishore, Oganesyan (2014)
  - Chandran, Kim, Vidal, Abanin (2015)
  - Ross, Mueller, Scardiccio (2014)
  - You, Qi, Xu (2015)

Diagonalization of the Hamiltonian

Local integrals of motion

We want to diagonalize the Hamiltonian

$$H = \sum_{\alpha} \phi_{\alpha} n_{\alpha} + \frac{1}{2} \sum_{\alpha\beta\gamma\eta} V_{\alpha\beta\gamma\eta} c_{\alpha}^{\dagger} c_{\beta} c_{\gamma}^{\dagger} c_{\eta}$$

The idea was to perform a basis change of the form

$$c_{\alpha} \rightarrow c_{\alpha} + \frac{V}{\Delta\phi} n_{\alpha} c_{\beta} c_{\gamma}^{\dagger} c_{\eta}$$

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Louk Rademaker (KITP) arXiv:1507.07276

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He found a family of transformations that diagonalize H in the form

$$H = \sum_{\alpha} b_{\alpha} \tilde{n}_{\alpha} + \sum_{\alpha\beta} b_{\alpha\beta} \tilde{n}_{\alpha} \tilde{n}_{\beta} + \cdots$$

obtaining explicitly the local integrals of motion

$$\tilde{n}_{\alpha} = U^{\dagger} n_{\alpha} U = n_{\alpha} + \sum_{\alpha\beta\gamma\eta} a_{\chi\beta\gamma\eta} c^{\dagger}_{\chi} c_{\beta} c^{\dagger}_{\gamma} c_{\eta} + \cdots$$

Displacement transformations

Displacement transformation: $\mathcal{D}_X(\lambda) = \exp\{\lambda(X^\dagger - X)\}$

$$X = n_{\alpha 1} \cdots n_{\alpha k} c_{\beta 1}^\dagger c_{\beta 2} c_{\beta 3}^\dagger \cdots c_{\beta l}$$

$$\mathcal{D}_X(\lambda) = 1 + \sin\lambda(X^\dagger - X) + (\cos\lambda - 1)(X^\dagger X + X X^\dagger)$$

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upper sign if $\delta = \beta$ or η and lower sign if $\delta = \alpha$ or γ

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$$\tan 2\lambda = \frac{V_{\alpha\beta\gamma\eta}}{\phi_\alpha + \phi_\gamma - \phi_\beta - \phi_\eta}$$

No problem with resonances

Trivial example

One particle in two sites

- $H = \phi_1 n_1 + \phi_2 n_2 - t(c_1^\dagger c_2 + c_2^\dagger c_1)$ $\qquad \begin{pmatrix} \phi_1 & -t \\ -t & \phi_2 \end{pmatrix}$
- Define $X = c_1^\dagger c_2$
- $\tan 2\lambda = \dfrac{-2t}{\phi_1 - \phi_2}$
- $\tilde{n}_1 = n_1 - \sin 2\lambda (X^\dagger + X) + \sin^2 \lambda [(1-n_1)n_2 - n_1(1-n_2)]$
- $\tilde{n}_2 = n_2 + \sin 2\lambda (X^\dagger + X) - \sin^2 \lambda [(1-n_1)n_2 - n_1(1-n_2)]$
- $\langle 10|\tilde{n}_1|10\rangle = 1 - \sin^2 \lambda$
- $\langle 10|\tilde{n}_2|10\rangle = \sin^2 \lambda$
- $\sin^2 \lambda = \dfrac{4t^2}{(\phi_1 - \phi_2)^2 + 4t^2}$

Consecutive transformations

- Each transformation modifies the remaining quantum terms of H.
  For example, for $X = c_\alpha^\dagger c_\beta c_\gamma^\dagger c_\eta$, $Y = c_\alpha^\dagger c_i c_\gamma^\dagger c_j$, $Z = c_\eta^\dagger c_i c_\beta^\dagger c_j$

$$\mathcal{D}_X^\dagger(\lambda)(Y^\dagger + Y)\mathcal{D}_X(\lambda) = \cos\lambda(Y^\dagger + Y) - \sin\lambda(Z^\dagger + Z) + \cdots$$

- One start with the transformation corresponding to the higher $|\lambda|$, i.e. higher $|V_{\alpha\beta\gamma\eta}/(\phi_\alpha + \phi_\gamma - \phi_\beta - \phi_\eta)|$, and continues performing consecutive transformations with decreasing values of λ until all four operators terms in H have been cancelled (to a certain accuracy).
- Continue transforming terms with 6 operators, then 8, and so on.
- Elliminating terms with a given number of operators do not generate terms with fewer operators.
- The final unitary transformation is

$$U = \prod_i \mathcal{D}_{X_i}(\lambda_i)$$

Practical implementation

CPU time $t \propto L^{3n/2}$

$V_{\alpha,\beta,\gamma,\eta}/(E_\alpha + E_\beta - E_\gamma - E_\eta)$

Time

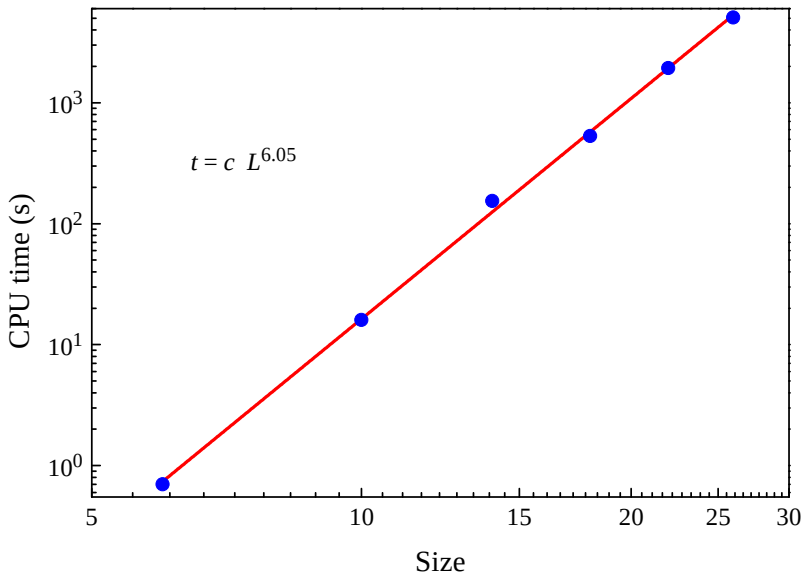

Figure: CPU time as a function of system size diagonalizing terms with 4 operators. $W = 6$ and cutoff equal 10^{-3}.

Layer by layer up to terms with 8 operators.

Strength

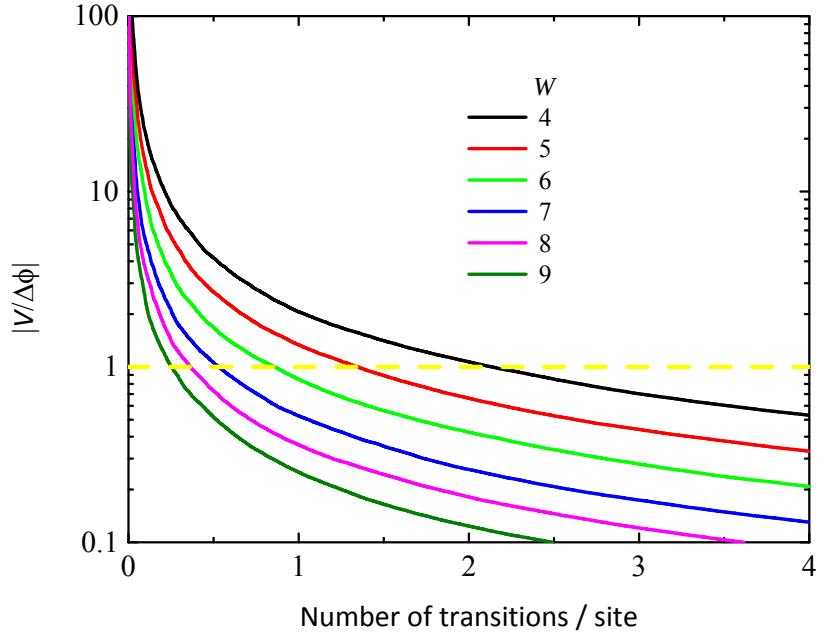

Figure: Number of transitions of strength $|V/\Delta\phi|$ for several disorders W.

Errors do not depend much on cutoff.

Hamiltonian in diagonal form

Coefficients

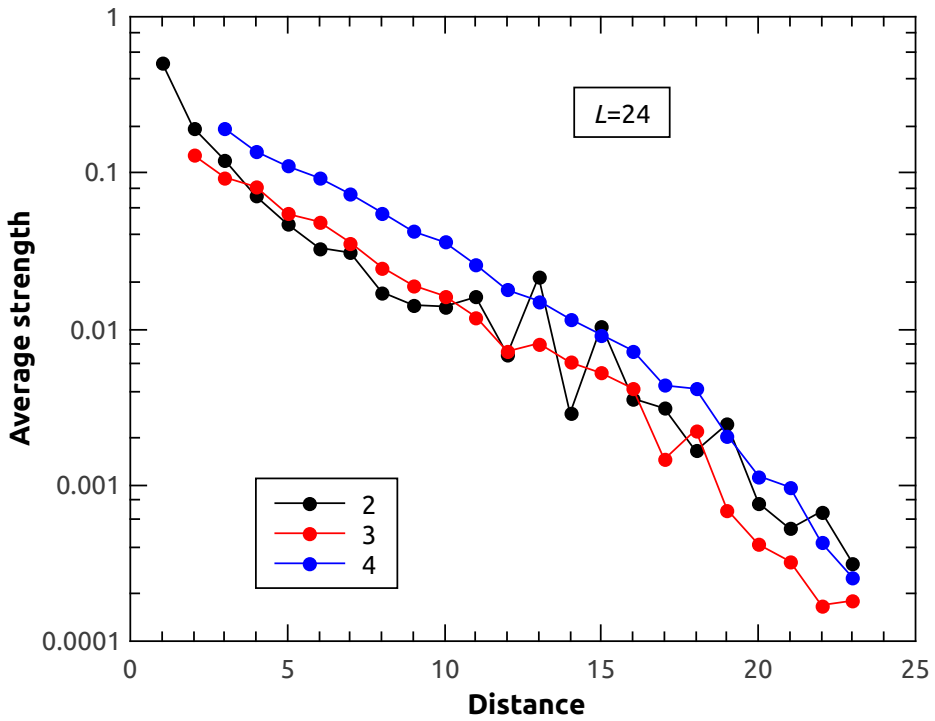

Figure: Average value of the coefficients of the Hamiltonian as a function of distance for $W = 8$.

$$H = \sum_{\alpha} b_{\alpha} \tilde{n}_{\alpha} + \sum_{\alpha\beta} b_{\alpha\beta} \tilde{n}_{\alpha} \tilde{n}_{\beta} + \cdots$$

- Structure of the coefficients: $\langle |b_{\alpha}| \rangle$, $\langle |b_{\alpha\beta}| \rangle$, $\ldots$
- Exponential decay
- Average value roughly independent of the number of operators n

Disorder dependence

Average coefficient

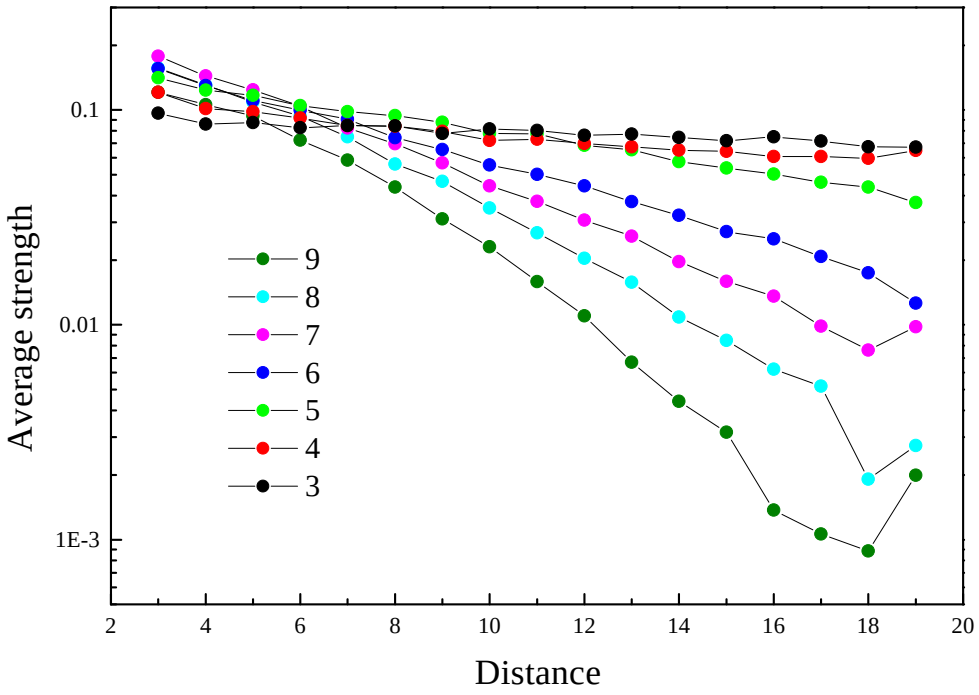

Figure: Average value of the coefficients of the Hamiltonian as a function of distance for several values of the disorder W.

- The slope gives us a disorder dependent localization length.
- In the extended phase the coefficients are independent of distance.
- The extended phase can be diagonalized with this method.

Distribution of the coefficients of H

(Unnormalized) distribution

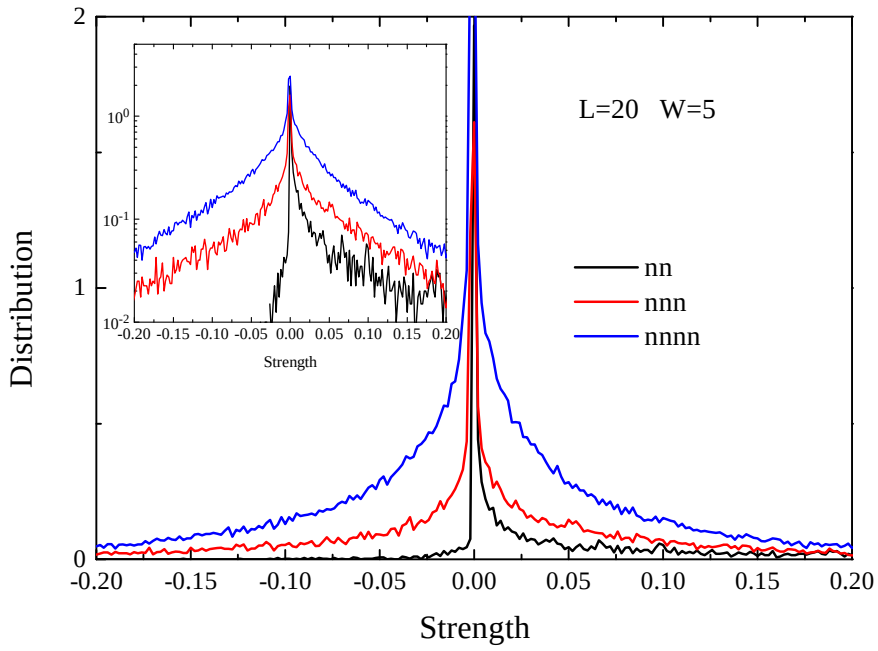

Figure: Unnormalized distribution function of different types of coefficients of H.

- Distribution for terms with two n operators is asymmetric. Net contribution.
- Contributions from higher order terms are roughly symmetric, but keep increasing.
- Choose the order of transformations that minimize higher order contributions. Or approximate high order terms by average values.

Errors

Ground state energy

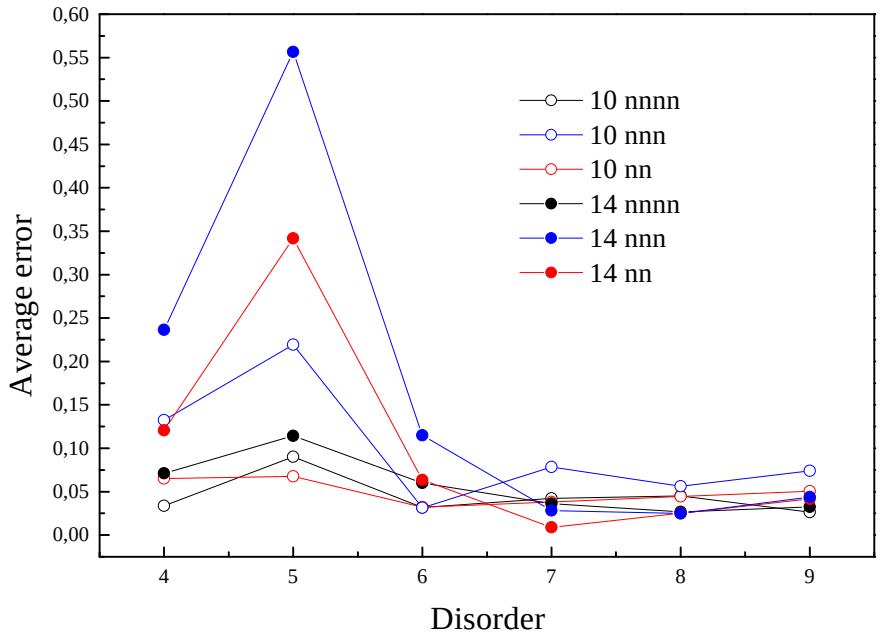

Figure: Error in the ground state energy as a function of disorder for two different sizes.

- The largest error are at the transition region.
- Errors depend on the number of terms considered, but not much on the cutoff for the coefficients.
- Relatively small contribution from high order terms.
- Error for the transmission amplitude are larger and, in this case, high order terms are crucial.

Integrals of motion

Exponential decay ($W = 8$)

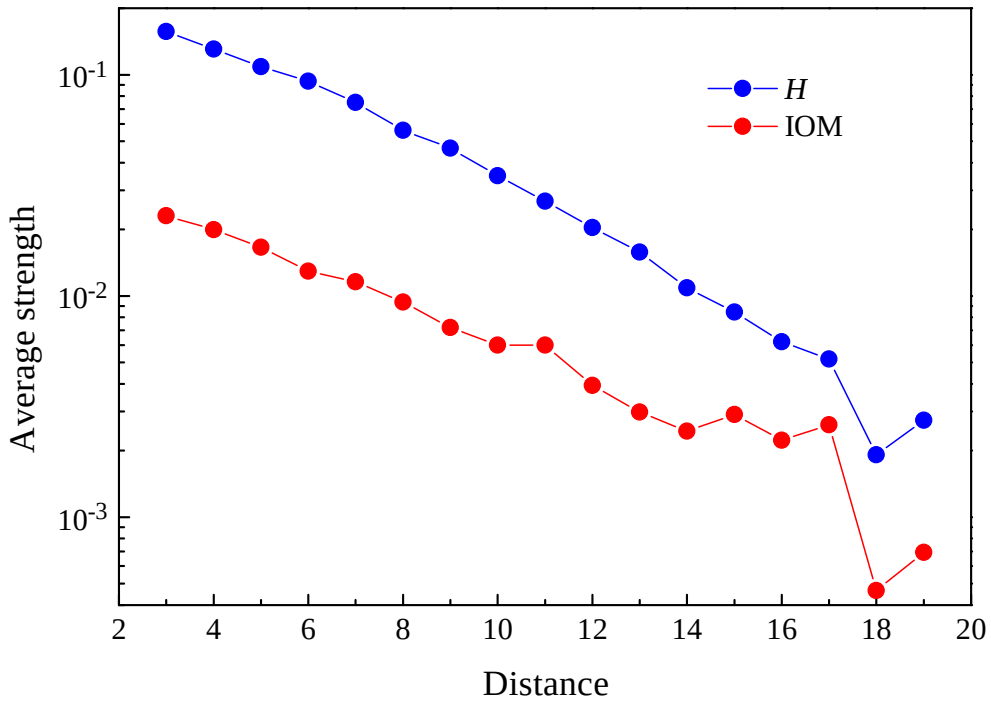

Figure: Comparisson between the exponential decay of the terms of H and of the classical terms of the IOM.

- Similar behavior between the terms of the Hamiltonian and the classical terms of the integrals of motion.
- Localization lengths slightly different.

Transmission amplitude

- Preliminary calculation of $\langle \ln |\tilde{c}_1^\dagger \tilde{c}_L| \rangle$ at infinite T.

Transmission

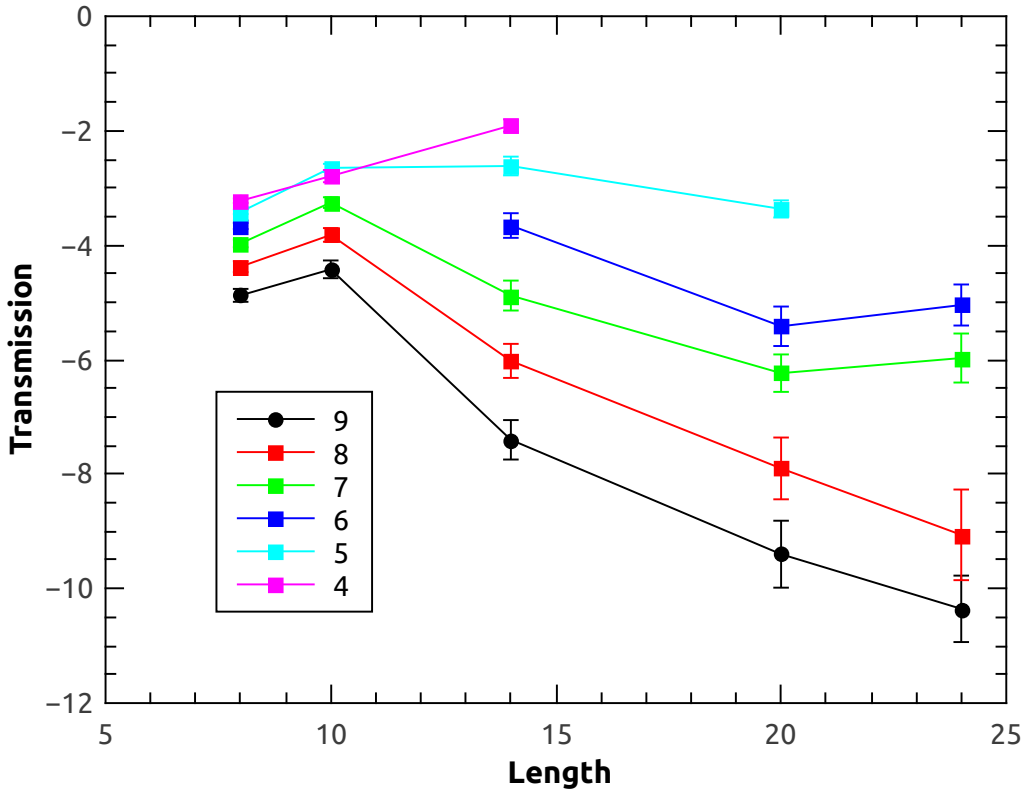

Figure: Transmission amplitude as a function of size for several values of the disorder.

Occupation number

- Simplified version.
- As a proof of principle, we calculate $\langle \Phi_0 | \tilde{n}_\alpha | \Phi_0 \rangle$ where $|\Phi_0\rangle = c^\dagger_\beta \cdots c^\dagger_\gamma |0\rangle$.
- There seems to be a transition at $W_c \approx 6$.

Occupation number

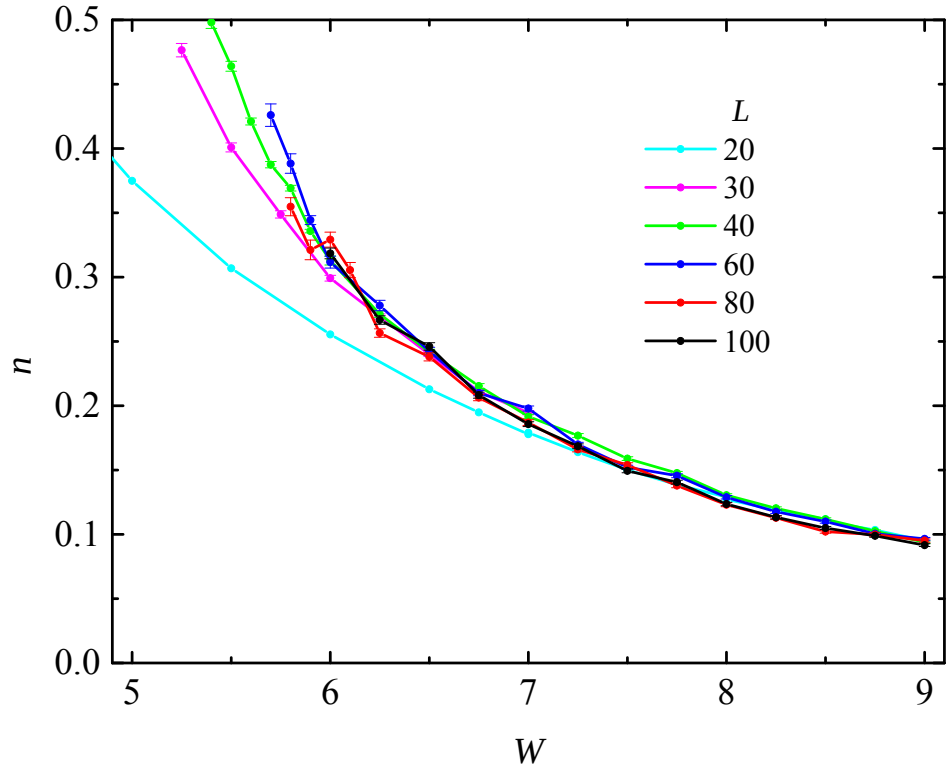

Figure: Average occupation number of a local integral of motion as a function of disorder.

Future aims and possibilities

- Study convergence of the method
- Best order of the transitions to minimize higher order terms
- Perform all transitions involving a site and then choose a value for its density operator.
- Quantum Coulomb gap (and probably Altshuler-Aronov gap)
- Level statistics